# MATH 54 – MOCK MIDTERM 2 – SOLUTIONS

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1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**.

(a) **FALSE** If $dim(V) = 3$ and $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $V$, then $\{\mathbf{u}, \mathbf{v}\}$ cannot be linearly independent!

(They *could* be linearly independent. For example, take $V = \mathbb{R}^3$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$! What *is* true, however, is that they cannot span $V$)

(b) **TRUE** If $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $T$ is onto, then $T$ is also one-to-one.

(This is the third miracle of Linear Algebra that I've been talking about! If you want to prove it, use the rank-nullity theorem!)

(c) **FALSE** If $A$ is a $m \times n$ matrix, then $Col(A)$ is a subspace of $\mathbb{R}^n$.

(It's a subspace of $\mathbb{R}^m$. For example, take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, which is a $2 \times 3$ matrix, then $Col(A) = Span \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^2$. In general, it's always good to write down an example of what $A$ looks like, so that you have an idea

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of what's going on!)

(d) **FALSE** If $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ is the change-of-coordinates matrix from $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ to $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$ then $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} [\mathbf{c_1}]_{\mathcal{B}} & [\mathbf{c_2}]_{\mathcal{B}} \end{bmatrix}$

(It's $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} [\mathbf{b_1}]_{\mathcal{C}} & [\mathbf{b_2}]_{\mathcal{C}} \end{bmatrix}$, you always take the old vectors (in $\mathcal{B}$) and evaluate them with respect to the new and *cool* basis $\mathcal{C}$)

(e) **TRUE** The Span of any set of vectors is always a vector space.

(see example 10 on page 209 for example)

2. *(20 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

(a) **FALSE** The set $V$ of $2 \times 2$ matrices such that $det(A) = 0$ is a vector space.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $det(A) = 0$ and $det(B) = 0$ so $A$ and $B$ are in $V$. But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $det(A+B) = 1 \neq 0$, so $A+B$ is not in $V$. Hence $V$ is not closed under addition, and hence is not a vector space.

(b) **TRUE** A $4 \times 5$ matrix $A$ cannot be invertible

**Hint:** How big is $Nul(A)$?

By the rank-nullity theorem, $dim(Nul(A)) + rank(A) = 5$. But $Rank(A) =$ number of pivots, which is at most $4$ (since $A$ has $4$ rows). Hence $dim(Nul(A)) \geq 5 - 4 = 1$. So $Nul(A) \neq \{\mathbf{0}\}$, hence $A$ is not invertible.

(c) **TRUE** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the set $V$ of $2 \times 2$ matrices $B$ such that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a vector space.

**Note:** By $O$, I mean $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

First of all, $V$ is a subset of $M_{2\times 2}$, the vector space of $2 \times 2$ matrices.

1) Zero-vector: $AO = O$, so the $O$-matrix is in $V$

2) Closed under addition: $B$ and $C$ are in $V$, then $AB = O$, and $AC = O$, so $A(B+C) = AB + AC = O + O = O$, so $B + C$ is in $V$

3) Closed under scalar multiplication: If $B$ is in $V$ and $c$ is in $\mathbb{R}$, $AB = O$, and so $A(cB) = cAB = c(O) = O$, so $cB$ in in $V$

Hence $V$ is a subspace of $M_{2\times 2}$ and hence is a vector space.

(d) **TRUE** The set $\{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ is a basis for $P_2$

First of all, identifying polynomials with a number code, we see that all we need to show is whether:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

Linear independence: To show that $\mathcal{B}$ is linearly independent, form the matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$. All we need to show is that $A\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. But if you row-reduce $A$, then you should get:

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & -5 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Which implies that $\mathbf{x} = \mathbf{0}$, hence $\mathcal{B}$ is linearly independent.

Span: Since $dim(\mathbb{R}^3) = 3$, and $\mathcal{B}$ is a linearly independent with $3$ vectors, we get that $\mathcal{B}$ spans $\mathbb{R}^3$ (this is one of the shortcuts I've been talking about in class).

Therefore $\mathcal{B}$ is a basis for $\mathbb{R}^3$, and hence $\{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ is a basis for $P_2$.

3. *(5 points)* Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points in $\mathbb{R}^2$ about the line $y = x$ and then rotates them by $180$ degrees ($\pi$ radians) counterclockwise.

We have:

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence the matrix of $T$ is:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

4. *(5 points)* A $2 \times 2$ matrix is called **symmetric** if $A^T = A$. Find a basis for the vector space $V$ of all $2 \times 2$ symmetric matrices. Show that the basis you found is in fact a basis!

**Hint:** What does a general $2 \times 2$ symmetric matrix look like?

A general $2 \times 2$ symmetric matrix has the form: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
Notice that:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We claim that:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is a basis for } V$$

Span: We just showed that! Any symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Linear Independence: (this part is **important**) Suppose:

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

And hence $a = 0, b = 0, c = 0$, and hence the set is linearly independent!

Therefore $\mathcal{B}$ is a basis for $V$ (and hence $V$ is 3-dimensional, but you didn't have to write this).

5. *(10 points)* For the following matrix $A$, find a basis for $Nul(A)$, $Row(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 6 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{Nul(A)}$ Since the right-hand-side is not in reduced row-echelon form, let's further row-reduce it:

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 9 & 0 & \frac{15}{2} \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(I first subtracted $3$ times the third row from the first row, and then added $\frac{1}{2}$ times the second row to the first row)

Now if $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix}$, then we get:

$$\begin{cases} 3x + 9z + \frac{15}{2}s = 0 \\ 2y + 4z + 3s = 0 \\ t + s = 0 \end{cases}$$

That is:

$$\begin{cases} x = -3z - \frac{5}{2}s \\ y = -2z - \frac{3}{2}s \\ t = -s \end{cases}$$

Hence we get:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3z - \frac{5}{2}s \\ -2z - \frac{3}{2}s \\ z \\ -s \\ s \end{bmatrix} = z \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

And therefore:

$$Nul(A) = Span \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\underline{Row(A)}$ Notice that there are pivots in the first, second, and third row, hence:

$$Row(A) = Span \left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\underline{Col(A)}$ Notice that there are pivots in the first, second, and fourth columns, hence:

$$Col(A) = Span \left\{ \begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix} \right\}$$

(Notice that you had to go back to the matrix $A$ to find a basis for $Col(A)$)

$\underline{Rank(A)}$ There are $3$ pivots, hence $Rank(A) = 3$.

6. *(10 points)* Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

   (a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, namely: $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$

$$[\mathcal{C} \mid \mathcal{B}] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

   (first I added $-4$ times the second row to the first, then I divided row 2 by $-3$, then I substracted the second row from the first row)

   Hence:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

   (b) Calculate $[\mathbf{x}]_{\mathcal{C}}$ given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

   We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B} \, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

7. *(10 points)* Let $V = Span\left\{e^x, e^x\cos(x), e^x\sin(x)\right\}$, and define $T : V \to V$ by:

$$T(y) = y' + y$$

   (a) Show $T$ is linear

$$T(y_1+y_2) = (y_1+y_2)'+(y_1+y_2) = (y_1)'+(y_2)'+y_1+y_2 = (y_1)'+y_1+(y_2)'+y_2 = T(y_1)+T(y_2)$$

$$T(cy) = (cy)' + cy = cy' + cy = c(y' + y) = cT(y)$$

Hence $T$ is a linear transformation.

(b) Find the matrix of $T$ with respect to the basis $\mathcal{B} = \{e^x, e^x\cos(x), e^x\sin(x)\}$ for $V$.

Again, don't freak out! For every vector/function in $\mathcal{B}$, evaluate $T$ of that function, and express your answer as a linear combination of the functions in $\mathcal{B}$.

$$\begin{aligned} T(e^x) =& (e^x)' + e^x \\ =& e^x + e^x \\ =& 2e^x \\ =& \mathbf{2}e^x + \mathbf{0}e^x\cos(x) + \mathbf{0}e^x\sin(x) \end{aligned}$$

$$\begin{aligned} T(e^x\cos(x)) =& (e^x\cos(x))' + e^x\cos(x) \\ =& e^x\cos(x) - e^x\sin(x) + e^x\cos(x) \\ =& 2e^x\cos(x) - e^x\sin(x) \\ =& \mathbf{0}e^x + \mathbf{2}e^x\cos(x) + (-\mathbf{1})e^x\sin(x) \end{aligned}$$

$$\begin{aligned} T(e^x\sin(x)) =& (e^x\sin(x))' + e^x\sin(x) \\ =& e^x\sin(x) + e^x\cos(x) + e^x\sin(x) \\ =& e^x\cos(x) + 2e^x\sin(x) \\ =& \mathbf{0}e^x + \mathbf{1}e^x\cos(x) + \mathbf{2}e^x\sin(x) \end{aligned}$$

Hence the matrix of $T$ is (just put the numbers in bold together **in columns**):

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

8. *(5 points)* Find the largest interval $(a, b)$ on which the following differential equation has a unique solution:

$$\sin(x)y'' + \left(\sqrt{2-x}\right)y' = e^x$$

with

$$y\left(\frac{\pi}{2}\right) = 4, y'\left(\frac{\pi}{2}\right) = 0$$

First convert the equation in standard form:

$$y'' + \left(\frac{\sqrt{2-x}}{\sin(x)}\right)y' = \frac{e^x}{\sin(x)}$$

Now let's look at the domain of each term:

The domain of $\frac{\sqrt{2-x}}{\sin(x)}$ is $(-\infty, 2] \cap \{x \neq n\pi\}$ (Basically $(-\infty, 2]$ without multiples of $\pi$, $\cap$ means 'intersection'). The part of that domain which contains the initial condition $\frac{\pi}{2}$ is $\boxed{(0, 2]}$

The domain of $\frac{e^x}{\sin(x)}$ is $\{x \neq n\pi\}$ (anything except multiples of $\pi$). The part of that domain which contains the initial condition $\frac{\pi}{2}$ is $\boxed{(0, \pi)}$

And if you intersect the two domains you found you get that the answer is $\boxed{(0, 2)}$.

**Note:** Make sure your answer is always an *open* interval! For example, here we got $(0, 2]$, but since it is not an open interval, we chose $(0, 2)$.

9. *(10 points)* Solve the following differential equation:

$$y''' - 12y'' + 41y' - 42y = 0$$

**Hint:** $42 = 2 \times 3 \times 7$.

The auxiliary equation is $r^3 - 12r^2 + 41r - 42 = 0$.

Now, by the rational roots theorem, we know that if the above polynomial has a rational root, then $r = \frac{a}{b}$, where $a$ divides the constant term $-42$ and $b$ divides the leading term $1$.

The only integers which divide $-42$ are (by the hint): $\pm 1, \pm 2, \pm 3, \pm 7, \pm 6, \pm 14, \pm 21, \pm 42$.

And the only integers which divide $1$ are $\pm 1$. Hence our guesses are: $\pm 1, \pm 2, \pm 3, \pm 7, \pm 6, \pm 14, \pm 21, \pm 42$.

If you plug-and-chug, you eventuall figure out that $r = 2$ works, i.e. $r = 2$ is a root of the auxiliary polynomial.

Now all you have to do is use long division and divide $r^3 - 12r^2 + 41r - 42$ by $r - 2$:

$$\begin{array}{r} X^2 - 10X + 21 \\ X - 2 \overline{\big) \; X^3 - 12X^2 + 41X - 42} \\ \underline{-X^3 + 2X^2} \\ -10X^2 + 41X \\ \underline{10X^2 - 20X} \\ 21X - 42 \\ \underline{-21X + 42} \\ 0 \end{array}$$

In other words, $r^3 - 12r^2 + 41r - 42 = (r^2 - 10r + 21)(r - 2) = (r-3)(r-7)(r-2) = (r-2)(r-3)(r-7)$

It follows that the roots are: $r = 2, 3, 7$, which means that the general solution is:

$$y(t) = Ae^{2t} + Be^{3t} + Ce^{7t}$$

**NOTE:** There are **NO** shortcuts to this problem! In particular, if you don't show the long division step on the exam, you will lose points!!!

10. *(10 points)*

(a) Solve $y'' + 4y' + 4y = e^{3t}$ using undetermined coefficients

Homogeneous solution: $r^2 + 4r + 4 = (r+2)^2 = 0$, which gives $r = -2$, a double root, hence $y_0(t) = Ae^{-2t} + Bte^{-2t}$.

Particular solution: Try $y_p = Ae^{3t}$. If you plug this into the differential equation, you get:

$$9Ae^{3t} + 12Ae^{3t} + 4Ae^{3t} = e^{3t} \Rightarrow 25A = 1 \Rightarrow A = \frac{1}{25}$$

So a particular solution is: $y_p(t) = \frac{1}{25}e^{3t}$

General solution:

$$y(t) = y_0(t) + y_p(t) = Ae^{-2t} + Bte^{-2t} + \frac{1}{25}e^{3t}$$

(b) Solve $y'' + y = \tan(t)$ using variation of parameters

**Note:** You may need to use the fact that $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Also you may use the fact that $\int \frac{\sin^2(t)}{\cos(t)}dt = \ln\left|\frac{\cos(t)}{\sin(t)-1}\right| - \sin(t)$.

Homogeneous solution: $r^2 + 1 = 0$, so $r = \pm i$, so $y_0(t) = A\cos(t) + B\sin(t)$

Particular solution: The Wronskian matrix is $\widetilde{W}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$.

Now $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$, where $v_1$ and $v_2$ solve:

$$\widetilde{W}(t)\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$$

That is:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \left(\widetilde{W}(t)\right)^{-1}\begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$$

But:

$$\left(\widetilde{W}(t)\right)^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}^{-1} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Hence:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -\sin(t)\tan(t) \\ \cos(t)\tan(t) \end{bmatrix}$$

That is:

$$v_1'(t) = -\sin(t)\tan(t) = -\sin(t)\left(\frac{\sin(t)}{\cos(t)}\right) = \frac{-\sin^2(t)}{\cos(t)}$$

So:

$$v_1(t) = -\int \frac{\sin^2(t)}{\cos(t)} = -\ln\left|\frac{\cos(t)}{\sin(t)-1}\right| + \sin(t)$$

(by the Hint, and ignore the constant)
And

$$v_2'(t) = \cos(t)\tan(t) = \cos(t)\left(\frac{\sin(t)}{\cos(t)}\right) = \sin(t)$$

So:

$$v_2(t) = -\cos(t)$$

Therefore:

$$\begin{aligned} y_p(t) =& v_1(t)\cos(t) + v_2(t)\sin(t) \\ =& \left(-\ln\left|\frac{\cos(t)}{\sin(t)-1}\right| + \sin(t)\right)\cos(t) - \cos(t)\sin(t) \\ =& -\ln\left|\frac{\cos(t)}{\sin(t)-1}\right|\cos(t) \end{aligned}$$

<u>General solution:</u>

$$y(t) = y_0(t) + y_p(t) = A\cos(t) + B\sin(t) - \ln\left|\frac{\cos(t)}{\sin(t)-1}\right|\cos(t)$$

11. *(5 points)* Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent vectors (in $V$) and $T : V \to W$ is a linear transformation. Show that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are also linearly dependent.

**Hint:** Write down what it means for $3$ vectors to be linearly dependent!

We know that for $a, b, c$ **not all** $0$:

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

Now apply $T$ to this equation:

$$T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = T(\mathbf{0}) = \mathbf{0}$$

But since $T$ is linear, we have:

$$aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$$

But since $a, b, c$ are not all $0$, this also means that the vectors $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are linearly dependent! And we're done!